

Geometric symmetries on Lorentzian manifolds

K. Saifullah*

*School of Mathematical Sciences, Queen Mary, University of London,
London, UK*

Electronic address: saifullah@qau.edu.pk

ABSTRACT: Lie derivatives of various geometrical and physical quantities define symmetries and conformal symmetries in general relativity. Thus we obtain motions, collineations, conformal motions and conformal collineations. These symmetries are used not only to find new solutions of Einstein's field equations but to classify the spaces also. Different classification schemes are presented here. Relationships between these symmetries are discussed and illustrating examples are presented.

* *On leave from:* Centre for Advanced Mathematics and Physics, National University of Sciences and Technology, Rawalpindi, Pakistan, *and* Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan.

1. Introduction

We consider a four dimensional Lorentzian manifold M with metric tensor g_{ab} , which is a function of the position given in coordinates by x^a , ($a, b, \dots = 0, 1, 2, 3$). If R_{ab} is the Ricci tensor and R is the Ricci scalar, Einstein's Field Equations (EFEs) can be written (without cosmological constant) as [1]

$$R_{ab} - \frac{1}{2}g_{ab}R = \kappa T_{ab}, \quad (1.1)$$

where T_{ab} is the energy-momentum tensor of the matter and κ is called the Einstein gravitational constant. These are the basic equations of the theory of general relativity (GR) which relate the geometry of the space to its matter content by virtue of which GR becomes the theory of the dependence of the metric of a Riemann manifold

$$ds^2 = g_{ab}dx^a dx^b, \quad (1.2)$$

on the distribution and motion of matter.

Since R_{ab} is a non-linear function of g_{ab} and its first and second derivatives, the EFEs are a system of 10 coupled highly non-linear second order partial differential equations for the 10 independent functions g_{ab} of four spacetime coordinates, x^a . Solving these equations amounts to determining the 10 components of the energy-momentum tensor, as well as the 10 components of g_{ab} of four variables; this makes the system undetermined. One way to tackle this problem is by making certain assumptions on the matter contents of the space (i.e. T_{ab}). On the other hand if no particular matter distribution is assumed, then the solutions can be obtained by imposing symmetry conditions on or by restricting the algebraic structure of the metric, the Ricci tensor or the Riemann tensor. Motions or Killing vectors (KVs), which are the symmetries of the metric, and Ricci collineations (RCs), which are the symmetries of the Ricci tensor, are two examples of such symmetries. These symmetry properties are described by continuous groups of motions or collineations and they lead to conservation laws. Apart from their significance in obtaining the exact solutions of the field equations, these symmetries provide various invariant bases for classification of spacetimes also. The techniques used for this purpose include application of groups of motions and algebraic classification of the Weyl tensor for what are called Petrov types [1] which were understood more simply by spinor methods developed by Penrose [2]. Segré classification [1] can also be obtained by spinors. Since the Ricci tensor and the energy-momentum tensor are mathematically similar, the study of RCs is important from the point of view of the study of symmetries of

matter (called matter collineations or MCs) also, apart from their geometrical significance. Other important symmetries in GR include homothetic motions (HMs), which are obtained when the Lie derivative of the metric is proportional to the metric; curvature and Weyl collineations — symmetries of the Riemann and Weyl tensors respectively. For an introduction to these symmetries in GR one can see Ref. 3.

In this paper we discuss different geometric symmetries in GR and various approaches to obtaining symmetries and classifying spacetimes. Relationships between these symmetries are also discussed and illustrating examples are presented.

2. Symmetries and the Lie derivative

‘The key to symmetries is the use of the Lie derivative’ [4]. For each point p in M , a vector field \mathbf{V} on M determines a unique curve $\alpha_p(t)$ such that $\alpha_p(0) = p$ and \mathbf{V} is the tangent vector to the curve. Now, consider a mapping h_t dragging each point p , with coordinates x^i , along the curve $\alpha_p(t)$ through p into the image point $q = h_t(p)$, with coordinates $y^i(t)$. If t is very small the map h_t is a one-one map and induces a map $h_t^*\mathbf{T}$ of any tensor \mathbf{T} . The Lie derivative of \mathbf{T} with respect to \mathbf{V} is defined by [1, 5]

$$\mathcal{L}_{\mathbf{V}}\mathbf{T} = \lim(t \rightarrow 0) \frac{1}{t} (h_t^*\mathbf{T} - \mathbf{T}). \quad (2.1)$$

Using the coordinate bases $\{\partial_{x^i}\}$ and $\{\partial_{y^i}\}$, the Lie derivative of a vector \mathbf{U} with respect to \mathbf{V} can be written as

$$\mathcal{L}_{\mathbf{V}}\mathbf{U} = v^m \frac{\partial}{\partial x^m} \left(u^i \frac{\partial}{\partial x^i} \right) - u^m \frac{\partial}{\partial x^m} \left(v^i \frac{\partial}{\partial x^i} \right) = [\mathbf{V}, \mathbf{U}], \quad (2.2)$$

where the commutator $[\mathbf{V}, \mathbf{U}]$ is the Lie bracket which is bilinear, antisymmetric and satisfies the Jacobi identity. A linear space of smooth vector fields under the operation of Lie bracket forms a Lie algebra. If $\{\mathbf{X}_i, i = 1, \dots, n\}$ is a basis for the Lie algebra, then we can always write

$$[\mathbf{X}_k, \mathbf{X}_l] = C_{kl}^j \mathbf{X}_j \quad C_{kl}^j = -C_{lk}^j. \quad (2.3)$$

Here C_{kl}^j are the structure constants which completely characterize the Lie algebra. If all the structure constants vanish the Lie algebra is Abelian. Every Lie algebra defines a unique simply connected Lie group and vice versa.

Lie derivatives are used in mathematical physics to express the invariance of a tensor field under some transformation. A tensor field \mathbf{T} is invariant under a vector

field \mathbf{V} if the tensors $h_t^*\mathbf{T}$ and \mathbf{T} coincide for t in some interval around 0, i.e., the Lie derivative vanishes

$$\mathcal{L}_{\mathbf{V}}\mathbf{T} = 0. \quad (2.4)$$

If \mathbf{T} has physical importance — e.g., it might be the metric tensor, or a scalar field describing a particle, or a vector field of force — then those special vector fields under which \mathbf{T} is invariant will also be important.

The manifolds of interest in mathematical physics have metric tensors. It is of interest to know when the metric is invariant with respect to some vector field. The vector fields along which the metric remains invariant are called Killing vector (KV) fields or isometries. After the spacetime metric, the curvature, Ricci and Weyl tensors are other important quantities that play a significant role in understanding the geometric structure of spacetimes in GR. While the isometries provide information of the symmetries inherent in the spacetime, the Ricci Collineations (RCs), vector fields along which the Ricci tensor is invariant under Lie transport, are important from the physical point of view as well.

3. Classification of spacetimes by symmetries

Here we formally define some of the symmetries used in general relativity and describe various classification schemes for the solutions of EFEs based on these symmetry methods.

3.1 Killing vectors

We call ξ a Killing vector or motion (or isometry) if the Lie derivative of \mathbf{g} with respect to ξ is conserved, i.e.

$$\mathcal{L}_{\xi}\mathbf{g} = 0. \quad (3.1)$$

This equation, called the Killing equation, in a torsion free space and in a coordinate basis, can also be written as

$$g_{ab,c}\xi^c + g_{ac}\xi_{,b}^c + g_{bc}\xi_{,a}^c = 0, \quad (3.2)$$

where “,” denotes the partial derivative. Its solutions are KVs. The set of all solutions of Eqs. (3.2) forms a Lie algebra and generates a Lie group of transformations. In a four dimensional space the maximum dimension of the Lie algebra is 10 [1]. Many explicit solutions of EFEs have been found using Killing symmetries. KVs can

be used to derive the most general axially symmetric stationary metric [6]. These symmetries leave all the curvature quantities invariant and they help in describing the kinematic and dynamic properties of spaces.

Soon after Killing's discovery of the Killing equations at the end of the nineteenth century and Lie's classification over the complex numbers of all Lie algebras up to dimension 3, Bianchi classified all 3 dimensional Riemannian manifolds according to their isometries. In this classification, which has recently been reprinted [7], Bianchi derived the full Killing vector Lie algebra for each possible symmetry class of group actions and solved the Killing equations to derive the metric. He also gave a representative line element for a given symmetry type.

The attempts to classify all solutions of the EFEs on the basis of KVs faced problems initially because of the arbitrariness of the energy-momentum tensor. A procedure was needed which could provide a list of all metrics according to a given isometry group and a complete list of all isometry groups. This is an alternate approach to solving the EFEs for given T_{ab} . Using this approach Eisenhart [8] succeeded in classifying all 2- and 3-dimensional spaces. He also developed important general theorems concerning groups of motions. Petrov [9] gave an invariant classification of Riemannian spacetimes admitting groups of motions on the basis of their detailed group structure. But his extension to 4-dimensional spaces was incomplete as admitted by him. However, Bokhari and Qadir [10] classified all static spherically symmetric spacetimes by solving the Killing equations for both g_{ab} and KVs. Obtaining the solutions for Eqs. (3.2) for the general spherically symmetric line element in the usual coordinates with ν and λ as arbitrary functions

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2 \quad (3.3)$$

gives a complete list of KVs for these spacetimes. The minimal dimension of the Lie algebra is 4 and maximum 10. The minimal symmetry is given by (writing $\frac{\partial}{\partial t}$ as ∂_t etc.)

$$\mathbf{X}_1 = \partial_t, \mathbf{X}_2 = \sin\phi\partial_\theta + \cos\phi\cot\theta\partial_\phi, \mathbf{X}_3 = \cos\phi\partial_\theta - \sin\phi\cot\theta\partial_\phi, \mathbf{X}_4 = \partial_\phi. \quad (3.4)$$

The maximum symmetry is admitted by the de Sitter, ant-de Sitter and Minkowski spaces. Qadir and Ziad [11] later provided a complete classification of *all* non-static spherically symmetric spacetimes according to their isometries and metrics. In this case the metric coefficients in the above metric were functions of t also. This study was extended to five-dimensional spaces [12] also.

Using these methods, classifications for static spacetimes with plane symmetry [13] and cylindrical symmetry [14] were also obtained. The stationary cylindrically symmetric fields are hypersurface-homogeneous spacetimes and admit three Killing vectors

$$\mathbf{X}_1 = \partial_t, \mathbf{X}_2 = \partial_\theta, \mathbf{X}_3 = \partial_z, \quad (3.5)$$

as the minimal symmetry which has the group $\mathbf{R} \otimes SO(2) \otimes \mathbf{R}$. The most general cylindrically symmetric static metric in coordinates (t, ρ, θ, z) can be written as [1]

$$ds^2 = e^{\nu(\rho)} dt^2 - d\rho^2 - a^2 e^{\lambda(\rho)} d\theta^2 - e^{\mu(\rho)} dz^2. \quad (3.6)$$

These spacetimes admit groups G_n of KVs, where $n = 3, 4, 5, 6, 7, 10$. For the plane symmetric metric

$$ds^2 = e^{\nu(t,x)} dt^2 - e^{\lambda(t,x)} dx^2 - e^{\mu(t,x)} (dy^2 + dz^2), \quad (3.7)$$

the minimal symmetry is given by

$$\mathbf{X}_1 = \partial_y, \mathbf{X}_2 = \partial_z, \mathbf{X}_3 = z\partial_y - y\partial_z. \quad (3.8)$$

In the static case the spacetimes admit a timelike KV, $\mathbf{X}_4 = \partial_t$, in addition to the KVs given above. The dimensions of the isometry group for the associated metrics are four, five, six, seven and ten; eight and nine are not admissible.

3.2 Homothetic motions

The vector ξ is said to be a homothetic motion (HM) if the right hand side of Eq. (3.1) is replaced by $\phi \mathbf{g}$, i.e.,

$$\mathcal{L}_\xi \mathbf{g} = \phi \mathbf{g}, \quad (3.9)$$

where ϕ is a nonzero constant.

As in the case of KVs, all the curvature quantities (except the scalar curvature which is preserved up to a constant factor) are invariant under a homothetic vector field. HMs give rise to self-similar spacetimes. It is known [8] that for the spaces admitting G_n as the maximal group of isometries, the group H_m for the HMs can be at most of the order $m = n + 1$.

Hall and Steele [15] investigated the Segré and Petrov types of spaces that admit proper homothety groups — HMs which are not KVs. They classified all such gravitational fields for homothety groups, H_m , $m \geq 6$, and gave some remarks for $m \leq 5$.

Hall [16] has also discussed the relation between homotheties and singularities of spacetimes.

Spherically symmetric spacetimes can have homothety groups [17] of the order $H_m, m = 4, 5, 6, 7, 8, 11$. For $r = 11$, the only spacetime is Minkowski. The classification of plane symmetric static [18] and the cylindrically symmetric static [19] Lorentzian manifolds according to their homotheties and metrics is consistent with the established theorems and known results. In the case of cylindrical symmetry some local homotheties can be extended globally also.

3.3 Ricci collineations

A vector field ξ is an RC if the Lie derivative of a Ricci tensor \mathbf{R} with respect to ξ is conserved, i.e.

$$\mathcal{L}_\xi \mathbf{R} = 0, \quad (3.10)$$

or in component form

$$\xi^c R_{ab,c} + R_{ac} \xi_{,b}^c + R_{bc} \xi_{,a}^c = 0. \quad (3.11)$$

There is one important point to note here that while the metric tensor is always non-degenerate (i.e. $\det \mathbf{g} \neq 0$) the other tensors like the Ricci and matter can be degenerate (i.e. the determinant is zero) also. This gives rise to the possibility of getting infinite dimensional algebras for the collineations.

Núñez *et al.* [20] studied RCs of the Robertson-Walker spacetime for which the complete solution was later provided by Camci and Barnes [21]. Melfo *et al.* [22] studied RC symmetry in Godel-type spacetimes and Hall *et al.* [23] studied RCs for various decomposable spacetimes and discussed the relationship between RCs and matter collineations. Camci *et al.* and other authors [24] discussed RCs for various Bianchi type spacetimes. An interesting study of RCs in type B warped spacetimes was done by Flores *et al.* [25]. RCs and MCs for locally rotationally symmetric spacetimes were studied by Tsamparlis and Apostolopoulos [26]. They studied these symmetries in Friedmann-Lemaitre universes also [27].

Bokhari and Qadir [28] and later Amir *et al.* [29] did some work on classification of static spherically symmetric spacetimes according to their RCs. Later Qadir and Ziad [30] and Contreras *et al.* [31] extended this study to the non-static case. Different aspects of RCs of this important class of spacetimes were also studied [32]. Plane symmetry may locally be thought of as a special case of cylindrical symmetry, therefore, the classification for plane symmetry [33] can be obtained as a special case

from the classification for cylindrical symmetry [34]. Some general observations from this work are:

1. For cylindrical symmetry the RC equations (3.11) are invariant under the interchange of any two of the three coordinates t , θ and z (indices 0, 2 and 3).
2. Minimal symmetry for cylindrically symmetric static spacetimes is given by $\langle \partial_t, \partial_\theta, \partial_z \rangle$, translations in t and z , and rotation in θ , and has the algebra $\mathbf{R} \otimes SO(2) \otimes \mathbf{R}$.
3. Cylindrically symmetric static spacetimes with non-degenerate Ricci tensor admit RCs with Lie algebras of dimensions 3, 4, 5, 6, 7 and 10. There are no 8- or 9-dimensional Lie algebras.
4. For the degenerate Ricci tensor the RCs have infinite dimensional Lie algebras except when $R_{11} = 0$ and R_{ii} ($i = 0, 2, 3$) are non-zero, the Lie algebras can be finite dimensional.
5. For the non-degenerate Ricci tensor, if any of the R_{00} , R_{22} or R_{33} components are nonzero constants, the space admits non-isometric RCs. This is an interesting and useful result for identifying proper RCs. No such result exists in the literature for the important class of spherically symmetric spacetimes and there is a need to do work in this direction.

As regards the physical significance of RCs, Davis *et al.* [35] did the pioneering work on the important role of RCs and the related conservation laws that are admitted by particular types of matter fields. They showed that the existence of isometries and collineations leads to conservation laws in the form of integrals of a dynamical system. They also considered the application of these results to relativistic hydrodynamics and plasma physics. Oliver and Davis [36] obtained conservation expressions for perfect fluids using RCs. The properties of fluid spacetimes admitting RCs were studied by Tsamparlis and Mason [37]. They have studied perfect fluid spacetimes in detail and have also considered a variety of imperfect fluids with cosmological constant and with anisotropic pressure.

3.4 Matter collineations

If the Ricci tensor in Eqs. (3.11) is replaced by the energy-momentum tensor then the vector field ξ is called a matter collineation (MC). Since the Ricci and matter

tensors are mathematically similar, the procedure for finding MCs is similar to that for RCs. Like KVs the maximum dimension of the Lie algebras for RCs and MCs is also ten in four dimensional space.

Carot *et al.* [38] provided one of the pioneering studies on this symmetry. Recently, MCs for different spacetimes have been discussed in the literature [39, 26, 27]. Various classes of spacetimes have been classified on the basis of this symmetry as well [40, 41]. Solving MC equations gives rise to various cases characterized by the constraints on the components of the energy-momentum tensor. This includes cases of non-degenerate as well degenerate tensors. Earlier remarks regarding RCs hold for MCs also.

3.5 Curvature collineations

Replacing \mathbf{g} by the Riemann curvature tensor, \mathbf{R} , in component form Eq. (3.1) becomes

$$R_{bcd,f}^a \xi^f + R_{fcd}^a \xi_{,b}^f + R_{bfd}^a \xi_{,c}^f + R_{bcf}^a \xi_{,d}^f - R_{bcd}^f \xi_{,f}^a = 0, \quad (3.12)$$

and the vector ξ is called a curvature collineation (CC). This is a more complicated system as compared to those of RCs or MCs, and is very tedious to solve completely. The maximum dimension of a CC algebra is also ten, when it is finite. CCs for different classes of spacetimes have been studied by Bokhari *et al.* [42], Hall and Shabbir [43] and Shabbir [44].

3.6 Weyl collineations

Replacing \mathbf{R} by the Weyl curvature tensor, \mathbf{C} , in Eq. (3.12) gives Weyl collineations (WCs) [45]

$$C_{bcd,f}^a \xi^f + C_{fcd}^a \xi_{,b}^f + C_{bfd}^a \xi_{,c}^f + C_{bcf}^a \xi_{,d}^f - C_{bcd}^f \xi_{,f}^a = 0. \quad (3.13)$$

Very little work [45, 46] has been done on this important symmetry and there is a need to do more in this direction.

4. Relationship between symmetries

It is clear from their definitions that motions, affine collineations (defined by $\mathcal{L}_\xi \Gamma_{bc}^a = 0$, where Γ_{bc}^a is the Christoffel symbol) and HMs are automatically CCs which are in turn RCs, but the converse is not true in general. The RCs (or MCs) which are not KVs are called proper RCs (or MCs) [34]. We note that RCs (and contracted Ricci collineations) are the most general of all the symmetries.

4.1 Isometries and other symmetries

Isometries or KVs are the fundamental symmetries in GR. Since the other tensors, like the Ricci and curvature tensors, are built from the metric tensor they must inherit its symmetries. Thus if the Lie derivative of \mathbf{g} vanishes, it must vanish for those tensors also. Hence every KV is an RC but the converse may not be true. We call the RCs which are not KVs “proper RCs”. For Einstein spaces, $\mathbf{R} \propto \mathbf{g}$, therefore, in this case the RCs and isometries coincide. Similarly, every KV is an MC, CC or WC but the converse is not true. As noted earlier the symmetries of the metric always have finite dimensional Lie algebras, which is not the case with those of the other tensors, in general.

4.2 Ricci and matter collineations

As mentioned earlier, if the components of the Ricci tensor, R_{ab} , in Eq. (3.11) are replaced by those of the energy-momentum tensor, we get MCs. But this does not mean that for a given space RCs and MCs will be identical. In some spaces RCs are greater than MCs, while in others MCs are more than RCs, which shows that neither of the sets contains the other, in general [40]. In some cases they may coincide as well. Let us consider the following metric for comparison of symmetries

$$ds^2 = (x/x_0)^{2a} dt^2 - dx^2 - (x/x_0)^2 (dy^2 + dz^2) ,$$

where a and x_0 are constants and $a \neq 0, 1, -1$. For this metric R_{ab} are given by

$$\begin{aligned} R_{00} &= a(1+a)x^{2a-2}/x_0^{2a} , \\ R_{11} &= -(-a+a^2)/x^2 , \\ R_{22} &= -(1+a)/x_0^2 = R_{33} . \end{aligned} \tag{4.1}$$

The energy-momentum tensor for this metric can be written as

$$\begin{aligned} T_{00} &= -x^{2a-2}/x_0^{2a} , \\ T_{11} &= (2a+1)/x^2 , \\ T_{22} &= a^2/x_0^2 = T_{33} . \end{aligned} \tag{4.2}$$

One may note that the energy density is negative and cannot be made positive by introducing a cosmological constant. The dimension of the KV algebra for this metric is four, while that of RCs and MCs is six, and it is spanned by

$$\begin{aligned}
\mathbf{X}_1 &= \partial_t, \\
\mathbf{X}_2 &= \partial_y, \\
\mathbf{X}_3 &= \partial_z, \\
\mathbf{X}_4 &= z\partial_y - y\partial_z, \\
\mathbf{X}_5 &= \left(\frac{1}{T_{00}} - \frac{1}{4}t^2\right)\partial_t + \frac{1}{\sqrt{T_{11}}}t\partial_x, \\
\mathbf{X}_6 &= -\frac{1}{2}t\partial_t + \frac{1}{\sqrt{T_{11}}}\partial_x.
\end{aligned} \tag{4.3}$$

The algebra is given by

$$\begin{aligned}
[\mathbf{X}_1, \mathbf{X}_5] &= \mathbf{X}_6, & [\mathbf{X}_1, \mathbf{X}_6] &= -\frac{1}{2}\mathbf{X}_1, & [\mathbf{X}_2, \mathbf{X}_4] &= -\mathbf{X}_3, \\
[\mathbf{X}_3, \mathbf{X}_4] &= \mathbf{X}_2, & [\mathbf{X}_5, \mathbf{X}_6] &= \frac{1}{2}\mathbf{X}_5, & [\mathbf{X}_i, \mathbf{X}_j] &= 0, \text{ otherwise.}
\end{aligned}$$

This is a direct sum of the two subalgebras $\{\mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4\}$ and $\{\mathbf{X}_1, \mathbf{X}_5, \mathbf{X}_6\}$.

4.3 Curvature and Weyl collineations

Though the Weyl and Riemann curvature tensors have similar forms, the local symmetries of the curvature tensor (i.e. CCs) and the Weyl tensor (WCs) are different. One can write a metric with a finite dimensional Lie algebra of Weyl symmetries that properly contains the Lie algebra of curvature symmetries [45]. However, there is no example known for the converse case.

Using the tensorial symmetries of the curvature or Weyl tensor, in component form they can be written as 6×6 matrices. Thus they are degenerate if they are of rank 5 or less. It is possible that one of them is degenerate and the other non-degenerate. An example is the de-Sitter (or anti de-Sitter) spacetime for which the Lie algebra of CCs is finite dimensional while that of WCs is infinite dimensional such that every vector field is a WC. The question arises whether there are cases in which the set of WCs is properly contained in the set of CCs when both have finite dimensional Lie algebras. So far, no such example exists in the literature; there is neither a proof that this cannot be the case. For the comparison of WCs and CCs we consider the metric [45]

$$ds^2 = dt^2 - d\rho^2 - (\rho/a)^2 d\theta^2 - (\rho/a)^2 dz^2. \tag{4.4}$$

In this case the stress-energy tensor is given by

$$T_{00} = -\frac{1}{\kappa\rho^2} = -T_{11}, T_{22} = 0 = T_{33},$$

so that it is not a physically realistic spacetime. The non-zero component of the curvature tensor is

$$R_{323}^2 = -\frac{1}{a^2}$$

and those of the Weyl tensor are

$$\begin{aligned} C_{101}^0 &= -\frac{1}{3\rho^2}, C_{202}^0 = \frac{1}{6} = C_{212}^1, \\ C_{303}^0 &= \frac{1}{6a^2} = C_{313}^1, C_{323}^2 = -\frac{1}{3a^2}. \end{aligned}$$

Here the Ricci tensor is degenerate. This space has one extra KV

$$\mathbf{X}_4 = -\frac{z}{a} \frac{\partial}{\partial \theta} + a\theta \frac{\partial}{\partial z},$$

one proper HM

$$\mathbf{X}_5 = t \frac{\partial}{\partial t} + \rho \frac{\partial}{\partial \rho},$$

and one additional WC

$$\mathbf{X}_6 = \frac{1}{2}(t^2 + \rho^2) \frac{\partial}{\partial t} + t\rho \frac{\partial}{\partial \rho}.$$

There are infinitely many CCs and RCs. Thus we see that in this case the set of KVs is contained in that of HMs which are a subset of WCs which are contained in CCs.

5. Summary and conclusion

Motions preserve distances. Conformal motions preserve angles between two directions at a point and map null geodesics into null geodesics. HMs scale all distances by the same constant factor, therefore, they lead to self-similar spacetimes. HMs also preserve the null geodesic affine parameters. Projective collineations

$$\mathcal{L}_\xi \Pi_{bc}^a = 0,$$

where the projective connection Π_{bc}^a is given by

$$\Pi_{bc}^a = \Gamma_{bc}^a - \frac{1}{n+1} (\delta_b^a \Gamma_{dc}^d + \delta_c^a \Gamma_{db}^d),$$

map geodesics into geodesics and affine collineations preserve, in addition, the affine parameters on geodesics [1].

New solutions of EFEs can be constructed by using symmetries. When the RC equations, for example, are solved to obtain the RC vectors, we also obtain some (differential) constraints on the Ricci tensor of the space. Solving these constraints gives the metric [34]. Another application of symmetries is that the solutions can be classified on their basis. We express symmetries in terms of Lie algebras which can be of finite or infinite dimensions.

Motions or KVs constitute the basic symmetry in GR. They form a subset of HMs which are contained in CCs. All these symmetries are subsets of RCs. A few symmetries have been described in some detail. The physical significance of KVs and HMs and their role in conservation laws is well understood. Similarly, WCs as the symmetries of the gravitational field and MCs of the matter field are important. However, the role of other symmetries from a physical point of view is yet to be understood [47]. Mutual relationship between some of these motions and collineations has been discussed. In particular, the inclusion relationships between RCs and MCs, and CCs and WCs have been explored. There is room for further research in this direction and some open problems have been mentioned.

We may mention here that besides these symmetries which have been described in this paper, there are other symmetries discussed in the literature [3, 47] on GR also.

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